Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Advances in Applied Mathematics 38 (2007) 267–274
ADVANCES IN Applied Mathematics
www.elsevier.com/locate/yaama

A connection between covers of the integers and unit fractions ☆

Zhi-Wei Sun

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

Received 8 July 2005; accepted 2 February 2006

Available online 28 August 2006

Abstract

For integers a and $n>0$, let $a(n)$ denote the residue class $\{x\in\mathbb{Z}\colon x\equiv a \pmod n\}$. Let A be a collection $\{a_s(n_s)\}_{s=1}^k$ of finitely many residue classes such that A covers all the integers at least m times but $\{a_s(n_s)\}_{s=1}^{k-1}$ does not. We show that if n_k is a period of the covering function $w_A(x)=|\{1\leqslant s\leqslant k\colon x\in a_s(n_s)\}|$ then for any $r=0,\ldots,n_k-1$ there are at least m integers in the form $\sum_{s\in I}1/n_s-r/n_k$ with $I\subseteq\{1,\ldots,k-1\}$.

MSC: primary 11B25; secondary 11B75, 11D68

1. Introduction

For an integer a and a positive integer n, we use $a(n)$ to denote the residue class $\{x\in\mathbb{Z}\colon x\equiv a \pmod n\}$. For a finite system

$$A=\big\{a_s(n_s)\big\}_{s=1}^k \tag{1}$$

of residue classes, the function $w_A:\mathbb{Z}\to\{0,1,\ldots\}$ given by

$$w_A(x)=\big|\big\{1\leqslant s\leqslant k\colon x\in a_s(n_s)\big\}\big|$$

☆ Supported by the National Science Fund for Distinguished Young Scholars (No. 10425103) and a Key Program of NSF (No. 10331020) in China.

E-mail address: zwsun@nju.edu.cn.
URL: http://pweb.nju.edu.cn/zwsun.

0196-8858/$ – see front matter
doi:10.1016/j.aam.2006.02.002

is called the *covering function* of A. Clearly $w_A(x)$ is periodic modulo the least common multiple N_A of the moduli $n_1, \ldots, n_k$, and it is easy to verify the following well-known equality:

$$\frac{1}{N_A}\sum_{x=0}^{N_A-1} w_A(x) = \sum_{s=1}^{k}\frac{1}{n_s}.$$

As in [7] we call $m(A) = \min_{x\in\mathbb{Z}} w_A(x)$ the *covering multiplicity* of A. For example,

$$B = \{0(2), 0(3), 1(4), 5(6), 7(12)\}$$

has covering multiplicity $m(B) = 1$, because the covering function is periodic modulo $N_B = 12$, and

$$w_B(x) = \begin{cases} 1 & \text{if } x \in \{1, 2, 3, 4, 7, 8, 10, 11\}, \\ 2 & \text{if } x \in \{0, 5, 6, 9\}. \end{cases}$$

Let m be a positive integer. If $w_A(x) \geqslant m$ for all $x \in \mathbb{Z}$, then we call A an *m-cover* of the integers, and in this case we have the well-known inequality $\sum_{s=1}^{k} 1/n_s \geqslant m$. (The term "1-cover" is usually replaced by the word "cover".) If A is an m-cover of the integers but $A_t = \{a_s(n_s)\}_{s\in[1,k]\setminus\{t\}}$ is not (where $[a, b] = \{x \in \mathbb{Z}:\ a \leqslant x \leqslant b\}$ for $a, b \in \mathbb{Z}$), then we say that A forms an m-cover of the integers with $a_t(n_t)$ *irredundant*. (For example, $\{0(2), 1(2), 2(3)\}$ is a cover of the integers in which $2(3)$ is redundant while $0(2)$ and $1(2)$ are irredundant.) If $w_A(x) = m$ for all $x \in \mathbb{Z}$, then A is said to be an *exact m-cover* of the integers, and in this case we have the equality $\sum_{s=1}^{k} 1/n_s = m$. By a graph-theoretic argument, M.Z. Zhang [12] showed that if $m > 1$ then there are infinitely many exact m-covers of the integers each of which cannot be split into an n-cover and an $(m - n)$-cover of the integers with $0 < n < m$.

Covers of the integers by residue classes were first introduced by P. Erdős (cf. [1]) in the 1930s, who observed that the system B mentioned above is a cover of the integers with distinct moduli. The topic of covers of the integers has been an active one in combinatorial number theory (cf. [3,4]), and many surprising applications have been found (see, e.g., [1,2,9,11]). The so-called m-covers and exact m-covers of the integers were systematically studied by the author in the 1990s.

Concerning the cover B given above one can easily check that

$$\left\{\sum_{n\in S}\frac{1}{n}:\ S \subseteq \{2, 3, 4, 6, 12\}\right\} = \left\{0, \frac{1}{12}, \ldots, \frac{11}{12}\right\} \cup \left\{1 + \frac{r}{12}:\ r = 0, 1, 2, 3, 4\right\}.$$

This suggests that for a general m-cover (1) of the integers we should investigate the set $\{\sum_{s\in I} 1/n_s:\ I \subseteq [1, k]\}$.

In this paper we establish the following new connection between covers of the integers and unit fractions.

Theorem 1. *Let $A = \{a_s(n_s)\}_{s=1}^{k}$ be an m-cover of the integers with the residue class $a_k(n_k)$ irredundant. If the covering function $w_A(x)$ is periodic modulo n_k, then for any $r = 0, \ldots, n_k - 1$ we have*

$$\left|\left\{\left\lfloor\sum_{s\in I}\frac{1}{n_s}\right\rfloor:\ I\subseteq[1,k-1]\ \textit{and}\ \left\{\sum_{s\in I}\frac{1}{n_s}\right\}=\frac{r}{n_k}\right\}\right|\geqslant m,\tag{2}$$

where $\lfloor\alpha\rfloor$ *and* $\{\alpha\}$ *denote the integral part and the fractional part of a real number* α*, respectively.*

Note that n_k in Theorem 1 need not be the largest modulus among $n_1,\dots,n_k$. In the case $m=1$ and $n_k=N_A$, Theorem 1 is an easy consequence of [5, Theorem 1] as observed by the author's twin brother Z.H. Sun. When $w_A(x)=m$ for all $x\in\mathbb{Z}$, the author [6] even proved the following stronger result:

$$\left|\left\{I\subseteq[1,k-1]:\ \sum_{s\in I}\frac{1}{n_s}=\frac{a}{n_k}\right\}\right|\geqslant\binom{m-1}{\lfloor a/n_k\rfloor}\quad\text{for all } a=0,1,\dots.$$

Given an m-cover $\{a_s(n_s)\}_{s=1}^k$ of the integers with $a_k(n_k)$ irredundant, by refining a result in [7] the author can show that there exists a real number $0\leqslant\alpha<1$ such that (2) with r/n_k replaced by $(\alpha+r)/n_k$ holds for every $r=0,\dots,n_k-1$.

Here we mention two local-global results related to Theorem 1.

(a) (Z.W. Sun [5]) $\{a_s(n_s)\}_{s=1}^k$ forms an m-cover of the integers if it covers $|\{\{\sum_{s\in I}1/n_s\}:\ I\subseteq[1,k]\}|$ consecutive integers at least m times.
(b) (Z.W. Sun [10]) $\{a_s(n_s)\}_{s=1}^k$ is an exact m-cover of the integers if it covers $|\bigcup_{s=1}^k\{r/n_s:\ r\in[0,n_s-1]\}|$ consecutive integers exactly m times.

Corollary 1. *Suppose that the covering function of* $A=\{a_s(n_s)\}_{s=1}^k$ *has a positive integer period* n_0*. If there is a unique* $a_0\in[0,n_0-1]$ *such that* $w_A(a_0)=m(A)$*, then for any* $D\subseteq\mathbb{Z}$ *with* $|D|=m(A)$ *we have*

$$\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:\ I\subseteq[1,k]\ \textit{and}\ \left\lfloor\sum_{s\in I}\frac{1}{n_s}\right\rfloor\notin D\right\}\supseteq\left\{\frac{r}{n_0}:\ r\in[0,n_0-1]\right\}.$$

Proof. Let $m=m(A)+1$. Clearly $A'=\{a_s(n_s)\}_{s=0}^k$ forms an m-cover of the integers with $a_0(n_0)$ irredundant. As $w_{A'}(x)-w_A(x)$ is the characteristic function of $a_0(n_0)$, $w_{A'}(x)$ is also periodic mod n_0. Applying Theorem 1 we immediately get the desired result. $\square$

We will prove Theorem 1 in Section 3 with help from some lemmas given in the next section.

2. Several lemmas

Lemma 1. *Let* (1) *be a finite system of residue classes with* $m(A)=m$*, and let* $m_1,\dots,m_k$ *be any integers. If* $f(x_1,\dots,x_k)$ *is a polynomial with coefficients in the complex field* $\mathbb{C}$ *and* $\deg f\leqslant m$*, then for any* $z\in\mathbb{Z}$ *we have*

$$\sum_{I\subseteq[1,k]}(-1)^{|I|}f\big([\![1\in I]\!],\dots,[\![k\in I]\!]\big)e^{2\pi i\sum_{s\in I}(a_s-z)m_s/n_s}$$
$$=(-1)^k c(I_z)\prod_{s\in[1,k]\setminus I_z}\big(e^{2\pi i(a_s-z)m_s/n_s}-1\big),\tag{3}$$

where $[\![s \in I]\!]$ *takes* 1 *or* 0 *according as* $s \in I$ *or not,* $I_z = \{1 \leqslant s \leqslant k\colon\ z \in a_s(n_s)\}$*, and* $c(I_z) = [\prod_{s\in I_z} x_s] f(x_1, \ldots, x_k)$ *is the coefficient of the monomial* $\prod_{s\in I_z} x_s$ *in* $f(x_1, \ldots, x_k)$.

Proof. Write $f(x_1, \ldots, x_k) = \sum_{j_1,\ldots,j_k \geqslant 0} c_{j_1,\ldots,j_k} x_1^{j_1} \cdots x_k^{j_k}$. Observe that

$$\begin{aligned}
&\sum_{I\subseteq[1,k]} (-1)^{|I|} f\big([\![1 \in I]\!], \ldots, [\![k \in I]\!]\big) e^{2\pi i \sum_{s\in I}(a_s - z)m_s/n_s} \\
&\quad= \sum_{\substack{j_1,\ldots,j_k \geqslant 0\\ j_1+\cdots+j_k \leqslant m}} c_{j_1,\ldots,j_k} \sum_{I\subseteq[1,k]} \left(\prod_{s=1}^{k} [\![s \in I]\!]^{j_s} \times (-1)^{|I|} e^{2\pi i \sum_{s\in I}(a_s - z)m_s/n_s} \right) \\
&\quad= \sum_{\substack{j_1,\ldots,j_k \geqslant 0\\ j_1+\cdots+j_k \leqslant m}} c_{j_1,\ldots,j_k} \sum_{J(j_1,\ldots,j_k)\subseteq I\subseteq[1,k]} (-1)^{|I|} e^{2\pi i \sum_{s\in I}(a_s - z)m_s/n_s},
\end{aligned}$$

where $J(j_1, \ldots, j_k) = \{1 \leqslant s \leqslant k\colon\ j_s \neq 0\}$.

Let $(j_1, \ldots, j_k)$ be nonnegative integers with $j_1 + \cdots + j_k \leqslant m$. If $I_z \nsubseteq J(j_1, \ldots, j_k)$, then

$$\sum_{J(j_1,\ldots,j_k)\subseteq I\subseteq[1,k]} (-1)^{|I|} e^{2\pi i \sum_{s\in I}(a_s - z)m_s/n_s} = 0$$

since

$$\begin{aligned}
&\sum_{I\subseteq[1,k]\setminus J(j_1,\ldots,j_k)} (-1)^{|I|} e^{2\pi i \sum_{s\in I}(a_s - z)m_s/n_s} \\
&\quad= \prod_{s\in[1,k]\setminus J(j_1,\ldots,j_k)} \big(1 - e^{2\pi i (a_s - z)m_s/n_s}\big) = 0.
\end{aligned}$$

If $I_z \subseteq J(j_1, \ldots, j_k)$, then

$$m = m(A) \leqslant |I_z| \leqslant \big|J(j_1, \ldots, j_k)\big| \leqslant j_1 + \cdots + j_k \leqslant m;$$

hence $I_z = J(j_1, \ldots, j_k)$ and $j_s = 1$ for all $s \in I_z$.

Combining the above we find that the left-hand side of (3) coincides with

$$\begin{aligned}
&c(I_z) \sum_{I_z\subseteq I\subseteq[1,k]} (-1)^{|I|} e^{2\pi i \sum_{s\in I}(a_s - z)m_s/n_s} \\
&\quad= c(I_z)(-1)^{|I_z|} e^{2\pi i \sum_{s\in I_z}(a_s - z)m_s/n_s} \prod_{s\in[1,k]\setminus I_z} \big(1 - e^{2\pi i (a_s - z)m_s/n_s}\big) \\
&\quad= (-1)^k c(I_z) \prod_{s\in[1,k]\setminus I_z} \big(e^{2\pi i (a_s - z)m_s/n_s} - 1\big).
\end{aligned}$$

This proves the desired equality (3). □

Lemma 2. *Let* (1) *be an m-cover of the integers with $a_k(n_k)$ irredundant, and let $m_1, \ldots, m_{k-1}$ be positive integers. Then, for any $0 \leqslant \alpha < 1$ we have $C_0(\alpha) = \cdots = C_{n_k-1}(\alpha)$, where $C_r(\alpha)$ (with $r \in [0, n_k - 1]$) denotes the sum*

$$\sum_{\substack{I\subseteq[1,k-1] \\ \{\sum_{s\in I} m_s/n_s\}=(\alpha+r)/n_k}} (-1)^{|I|} \binom{\lfloor \sum_{s\in I} m_s/n_s \rfloor}{m-1} e^{2\pi i \sum_{s\in I}(a_s-a_k)m_s/n_s}.$$

Proof. This follows from [7, Lemma 2]. □

Lemma 3. *Let* (1) *be an m-cover of the integers with $a_k(n_k)$ irredundant. Suppose that n_k is a period of the covering function $w_A(x)$. Then, for any $z \in a_k(n_k)$ we have*

$$\prod_{s\in[1,k]\setminus I_z} \left(1 - e^{2\pi i(a_s-z)/n_s}\right) = \prod_{s\in I_z} n_s \times \prod_{t=1}^{n_k}\left(1 - e^{2\pi i(t-a_k)/n_k}\right)^{w_A(t)-m},$$

where $I_z = \{1 \leqslant s \leqslant k\colon\ z \in a_s(n_s)\}$.

Proof. Since $a_k(n_k)$ is irredundant, we have $w_A(z_0) = m$ for some $z_0 \in a_k(n_k)$. As the covering function of A is periodic mod n_k, $|I_z| = w_A(z) = m$ for all $z \in a_k(n_k)$.

Now fix $z \in a_k(n_k)$. Since $w_A(x)$ is periodic modulo n_k, by [8, Lemma 2.1] we have the identity

$$\prod_{s=1}^{k}\left(1 - y^{N/n_s} e^{2\pi i a_s/n_s}\right) = \prod_{t=1}^{n_k}\left(1 - y^{N/n_k} e^{2\pi i t/n_k}\right)^{w_A(t)},$$

where $N = N_A$ is the least common multiple of $n_1, \ldots, n_k$. Putting $y = r^{1/N} e^{-2\pi i z/N}$ with $r \geqslant 0$, we then get that

$$\prod_{s=1}^{k}\left(1 - r^{1/n_s} e^{2\pi i(a_s-z)/n_s}\right) = \prod_{t=1}^{n_k}\left(1 - r^{1/n_k} e^{2\pi i(t-z)/n_k}\right)^{w_A(t)}.$$

Therefore

$$\begin{aligned}
&\prod_{s\in[1,k]\setminus I_z} \left(1 - e^{2\pi i(a_s-z)/n_s}\right) \\
&\quad = \lim_{r\to 1} \prod_{s\in[1,k]\setminus I_z} \left(1 - r^{1/n_s} e^{2\pi i(a_s-z)/n_s}\right) \\
&\quad = \lim_{r\to 1} \frac{\prod_{t=1}^{n_k}(1 - r^{1/n_k} e^{2\pi i(t-a_k)/n_k})^{w_A(t)}}{\prod_{s\in I_z}(1 - r^{1/n_s} e^{2\pi i(a_s-z)/n_s})}
\end{aligned}$$

$$= \lim_{r\to 1}\prod_{s\in I_z}\frac{1-r}{1-r^{1/n_s}} \times \lim_{r\to 1}\frac{\prod_{t=1}^{n_k}(1-r^{1/n_k}e^{2\pi i(t-a_k)/n_k})^{w_A(t)}}{(1-r)^m}$$

$$= \prod_{s\in I_z} n_s \times \lim_{r\to 1}\prod_{t=1}^{n_k}\left(1-r^{1/n_k}e^{2\pi i(t-a_k)/n_k}\right)^{w_A(t)-m},$$

and hence the desired result follows. □

3. Proof of Theorem 1

In the case $k=1$, we must have $m=1$ and $n_k=1$; hence the required result is trivial. Below we assume that $k>1$.

Let $r_0 \in [0, n_k-1]$ and $D=\{d_n + r_0/n_k\colon\ n\in[1,m-1]\}$, where $d_1,\dots,d_{m-1}$ are $m-1$ distinct nonnegative integers. (If $m=1$ then we set $D=\emptyset$.) We want to show that there exists $I\subseteq[1,k-1]$ such that $\{\sum_{s\in I}1/n_s\} = r_0/n_k$ and $\sum_{s\in I}1/n_s \notin D$.

Define

$$f(x_1,\dots,x_{k-1}) = \prod_{d\in D}\left(\frac{x_1}{n_1}+\dots+\frac{x_{k-1}}{n_{k-1}}-d\right).$$

(An empty product is regarded as 1.) Then $\deg f = |D| = m-1$. For any $z\in a_k(n_k)$, the set $I_z=\{1\leqslant s\leqslant k\colon\ z\in a_s(n_s)\}$ has cardinality m since $a_k(n_k)$ is irredundant and $w_A(x)$ is periodic mod n_k. Observe that the coefficient

$$c_z = \left[\prod_{s\in I_z\setminus\{k\}} x_s\right] f(x_1,\dots,x_{k-1}) = \left[\prod_{s\in I_z\setminus\{k\}} x_s\right]\left(\sum_{s=1}^{k-1}\frac{x_s}{n_s}\right)^{m-1}$$

coincides with $(m-1)!/\prod_{s\in I_z\setminus\{k\}} n_s$ by the multinomial theorem. For $I\subseteq[1,k-1]$ we set

$$v(I) = f\big([\![1\in I]\!],\dots,[\![k-1\in I]\!]\big).$$

As $|\{1\leqslant s\leqslant k-1\colon\ x\in a_s(n_s)\}| \geqslant \deg f$ for all $x\in\mathbb{Z}$, in view of Lemmas 1 and 3 we have

$$\begin{aligned}
&\sum_{I\subseteq[1,k-1]}(-1)^{|I|}v(I)e^{2\pi i\sum_{s\in I}(a_s-z)/n_s}\\
&= (-1)^{k-1}c_z \prod_{s\in[1,k-1]\setminus I_z}\left(e^{2\pi i(a_s-z)/n_s}-1\right)\\
&= \frac{(-1)^{m-1}(m-1)!}{\prod_{s\in I_z\setminus\{k\}}n_s}\prod_{s\in I_z}n_s \times \prod_{t=1}^{n_k}\left(1-e^{2\pi i(t-a_k)/n_k}\right)^{w_A(t)-m} = C,
\end{aligned}$$

where C is a nonzero constant not depending on $z\in a_k(n_k)$.

By the above,

$$N_A C = \sum_{x=0}^{N_A-1} \sum_{I\subseteq[1,k-1]} (-1)^{|I|} v(I) e^{2\pi i \sum_{s\in I}(a_s-a_k-n_k x)/n_s}$$
$$= \sum_{I\subseteq[1,k-1]} (-1)^{|I|} v(I) e^{2\pi i \sum_{s\in I}(a_s-a_k)/n_s} \sum_{x=0}^{N_A-1} e^{-2\pi i x \sum_{s\in I} n_k/n_s},$$

and hence

$$C = \sum_{\substack{I\subseteq[1,k-1]\\ n_k\sum_{s\in I}1/n_s\in\mathbb{Z}}} (-1)^{|I|} v(I) e^{2\pi i \sum_{s\in I}(a_s-a_k)/n_s} = \sum_{r=0}^{n_k-1} C_r,$$

where

$$C_r = \sum_{\substack{I\subseteq[1,k-1]\\ \{\sum_{s\in I}1/n_s\}=r/n_k}} (-1)^{|I|} \prod_{d\in D}\left(\sum_{s\in I}\frac{1}{n_s} - d\right) e^{2\pi i \sum_{s\in I}(a_s-a_k)/n_s}.$$

Let $r\in[0,n_k-1]$. Write

$$P_r(x) = \prod_{d\in D}\left(x+\frac{r}{n_k}-d\right) = \sum_{n=0}^{m-1} c_{n,r}\binom{x}{n},$$

where $c_{n,r}\in\mathbb{C}$. By comparing the leading coefficients, we find that $c_{m-1,r}=(m-1)!$. Observe that

$$C_r = \sum_{\substack{I\subseteq[1,k-1]\\ \{\sum_{s\in I}1/n_s\}=r/n_k}} (-1)^{|I|} P_r\left(\left\lfloor \sum_{s\in I}\frac{1}{n_s}\right\rfloor\right) e^{2\pi i \sum_{s\in I}(a_s-a_k)/n_s}$$
$$= \sum_{n=0}^{m-1} c_{n,r} \sum_{\substack{I\subseteq[1,k-1]\\ \{\sum_{s\in I}1/n_s\}=r/n_k}} (-1)^{|I|} \binom{\lfloor\sum_{s\in I}1/n_s\rfloor}{n} e^{2\pi i \sum_{s\in I}(a_s-a_k)/n_s}$$
$$= c_{m-1,r} \sum_{\substack{I\subseteq[1,k-1]\\ \{\sum_{s\in I}1/n_s\}=r/n_k}} (-1)^{|I|} \binom{\lfloor\sum_{s\in I}1/n_s\rfloor}{m-1} e^{2\pi i \sum_{s\in I}(a_s-a_k)/n_s};$$

in taking the last step we note that if $0\leqslant n<m-1$ then

$$\sum_{\substack{I\subseteq[1,k-1]\\ \{\sum_{s\in I}1/n_s\}=r/n_k}} (-1)^{|I|} \binom{\lfloor\sum_{s\in I}1/n_s\rfloor}{n} e^{2\pi i \sum_{s\in I}a_s/n_s} = 0$$

by [5, Theorem 1] (since $\{a_s(n_s)\}_{s=1}^{k-1}$ is an $(m-1)$-cover of the integers). By Lemma 2 and the above,

$$C_r = (m-1)! \sum_{\substack{I\subseteq[1,k-1]\\ \{\sum_{s\in I}1/n_s\}=0}} (-1)^{|I|}\binom{\lfloor\sum_{s\in I}1/n_s\rfloor}{m-1}e^{2\pi i\sum_{s\in I}(a_s-a_k)/n_s}$$

does not depend on $r\in[0,n_k-1]$.

Combining the above we obtain that

$$n_kC_{r_0} = \sum_{r=0}^{n_k-1} C_r = C \neq 0.$$

So there is $I\subseteq[1,k-1]$ for which $\{\sum_{s\in I}1/n_s\} = r_0/n_k$, $\sum_{s\in I}1/n_s \notin D$ and hence $\lfloor\sum_{s\in I}1/n_s\rfloor \notin \{d_n\colon n\in[1,m-1]\}$. This concludes our proof.

References

[1] P. Erdős, On integers of the form 2^k+p and some related problems, Summa Brasil. Math. 2 (1950) 113–123.
[2] M. Filaseta, Coverings of the integers associated with an irreducibility theorem of A. Schinzel, in: M.A. Bennett, B.C. Berndt, N. Boston, H.G. Diamond, A.J. Hildebrand, W. Philipp (Eds.), Number Theory for the Millennium, vol. II, Urbana, IL, 2000, A.K. Peters, Natick, MA, 2002, pp. 1–24.
[3] R.K. Guy, Unsolved Problems in Number Theory, third ed., Springer, New York, 2004 (Sections F13 and F14).
[4] Š. Porubský, J. Schönheim, Covering systems of Paul Erdös: Past, present and future, in: G. Halász, L. Lovász, M. Simonvits, V.T. Sós (Eds.), Paul Erdös and His Mathematics, I, Bolyai Soc. Math. Stud., vol. 11, 2002, pp. 581–627.
[5] Z.W. Sun, Covering the integers by arithmetic sequences, Acta Arith. 72 (1995) 109–129.
[6] Z.W. Sun, Exact m-covers and the linear form $\sum_{s=1}^k x_s/n_s$, Acta Arith. 81 (1997) 175–198.
[7] Z.W. Sun, On covering multiplicity, Proc. Amer. Math. Soc. 127 (1999) 1293–1300.
[8] Z.W. Sun, On the function $w(x)=|\{1\leqslant s\leqslant k\colon x\equiv a_s \pmod{n_s}\}|$, Combinatorica 23 (2003) 681–691.
[9] Z.W. Sun, Unification of zero-sum problems, subset sums and covers of $\mathbb{Z}$, Electron. Res. Announc. Amer. Math. Soc. 9 (2003) 51–60.
[10] Z.W. Sun, Arithmetic properties of periodic maps, Math. Res. Lett. 11 (2004) 187–196.
[11] Z.W. Sun, A local-global theorem on periodic maps, J. Algebra 293 (2005) 506–512.
[12] M.Z. Zhang, Irreducible systems of residue classes that cover every integer exactly m times, Sichuan Daxue Xuebao (Nat. Sci. Ed.) 28 (1991) 403–408.